

Invariant functions on p -divisible groups and the p -adic Corona problem

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1 Introduction

In this note we are concerned with p -divisible groups $G = (G_\nu)$ over a complete discrete valuation ring R . We assume that the fraction field K of R has characteristic zero and that the residue field $k = R/\pi R$ is perfect of positive characteristic p .

Let C be the completion of an algebraic closure of K and denote by $\mathfrak{o} = \mathfrak{o}_C$ its ring of integers. The group $G_\nu(\mathfrak{o})$ acts on $G_\nu \otimes \mathfrak{o}$ by translation. Since $G_\nu \otimes K$ is étale the $G_\nu(C)$ -invariant functions on $G_\nu \otimes C$ are just the constants. Using the counit it follows that the natural inclusion

$$\mathfrak{o} \xrightarrow{\sim} \Gamma(G_\nu \otimes \mathfrak{o}, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is an isomorphism. We are interested in an approximate mod π^n -version of this statement. Set $\mathfrak{o}_n = \mathfrak{o}/\pi^n \mathfrak{o}$ for $n \geq 1$. The group $G_\nu(\mathfrak{o})$ acts by translation on $G_\nu \otimes \mathfrak{o}_n$ for all n .

Theorem 1 *Assume that the dual p -divisible group G' is at most one-dimensional and that the connected-étale exact sequence for G' splits over \mathfrak{o} . Then there is an integer $t \geq 1$ such that the cokernel of the natural inclusion*

$$\mathfrak{o}_n \hookrightarrow \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})^{G_\nu(\mathfrak{o})}$$

is annihilated by p^t for all ν and n .

The example of $\mathbb{G}_m = (\mu_{p^\nu})$ in section 2 may be helpful to get a feeling for the statement.

We expect the theorem to hold without any restriction on the dimension of G as will be explained later. Its assertion is somewhat technical but the proof may be of interest because it combines some of the main results of Tate on p -divisible groups with van der Put's solution of his one-dimensional p -adic Corona problem.

The classic corona problem concerns the Banach algebra $H^\infty(D)$ of bounded analytic functions on the open unit disc D . The points of D give maximal ideals in $H^\infty(D)$ and hence points of the Gelfand spectrum $\hat{D} = \text{sp } H^\infty(D)$. The question was whether D was dense in \hat{D} , (the set $\hat{D} \setminus \overline{D}$ being the “corona”). This was settled affirmatively by Carleson [C]. The analogous question for the polydisc D^d is still open for $d \geq 2$. An equivalent condition for D^d to be dense in $\text{sp } H^\infty(D^d)$ is the following one, [H], Ch. 10:

Condition 2 *If f_1, \dots, f_n are bounded analytic functions in D^d such that for some $\delta > 0$ we have*

$$\max_{1 \leq i \leq n} |f_i(z)| \geq \delta \quad \text{for all } z \in D^d ,$$

then f_1, \dots, f_n generate the unit ideal of $H^\infty(D^d)$.

In [P] van der Put considered the analogue of condition 2 with $H^\infty(D^d)$ replaced by the algebra of bounded analytic C -valued functions on the p -adic open polydisc Δ^d in C^d , i.e. by the algebra

$$C\langle X_1, \dots, X_d \rangle = \mathfrak{o}[[X_1, \dots, X_d]] \otimes_{\mathfrak{o}} C .$$

He called this p -adic version of condition 2 the p -adic Corona problem and verified it for $d = 1$. The general case $d \geq 1$ was later treated by Bartenwerfer [B] using his earlier results on rigid cohomology with bounds.

In the proof of theorem 1 applying Tate's results from [T] we are led to a question about certain ideals in $C\langle X_1, \dots, X_d \rangle$, which for $d = 1$ can be reduced to van der Put's p -adic Corona problem. For $d \geq 2$, I did not succeed in such a reduction. However it seems possible that a generalization of Bartenwerfer's theory might settle that question.

It should be mentioned that van der Put's term " p -adic Corona problem" for the p -adic analogue of condition 2 is somewhat misleading. Namely as pointed out in [EM] a more natural analogue would be the question whether Δ^d was dense in the Berkovich space of $C\langle X_1, \dots, X_d \rangle$. This is not known, even for $d = 1$. The difference between the classic and the p -adic cases comes from the fact discovered by van der Put that contrary to $H^\infty(D^d)$ the algebra $C\langle X_1, \dots, X_d \rangle$ contains maximal ideals of infinite codimension.

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2 An example and other versions of the theorem

Consider an affine group scheme \mathcal{G} over a ring S with Hopf-algebra $\mathcal{A} = \Gamma(\mathcal{G}, \mathcal{O})$, comultiplication $\mu : \mathcal{A} \rightarrow \mathcal{A} \otimes_S \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \rightarrow S$. The operation of $\mathcal{G}(S) = \text{Hom}_S(\mathcal{A}, S)$ on $\Gamma(\mathcal{G}, \mathcal{O})$ by translation is given by the map

$$(1) \quad \mathcal{G}(S) \times \mathcal{A} \rightarrow \mathcal{A}, (\chi, a) \mapsto (\chi \otimes \text{id})\mu(a)$$

where $(\chi \otimes \text{id})\mu$ is the composition

$$\mathcal{A} \xrightarrow{\mu} \mathcal{A} \otimes_S \mathcal{A} \xrightarrow{\chi \otimes \text{id}} S \otimes_S \mathcal{A} = \mathcal{A}.$$

Given a homomorphism of groups $P \rightarrow \mathcal{G}(S)$ we may view \mathcal{A} as a P -module. The composition $S \rightarrow \mathcal{A} \xrightarrow{\varepsilon} S$ being the identity we have an isomorphism

$$(2) \quad \text{Ker}(\mathcal{A}^P \xrightarrow{\varepsilon} S) \xrightarrow{\sim} \mathcal{A}^P / S \quad \text{mapping } a \text{ to } a + S.$$

The inverse sends $a + S$ to $a - \varepsilon(a) \cdot 1$.

Example The theorem is true for $\mathbb{G}_m = (\mu_{p^\nu})$.

Proof Set $V = \mathfrak{o}_n[X, X^{-1}]/(X^{p^\nu} - 1)$. Applying formulas (1) and (2) with $\mathcal{G} = \mu_{p^\nu} \otimes \mathfrak{o}_n$ and $P = \mu_{p^\nu}(\mathfrak{o}) \rightarrow \mathcal{G}(\mathfrak{o}_n)$ we see that the cokernel of the map

$$(3) \quad \mathfrak{o}_n \rightarrow \Gamma(\mu_{p^\nu} \otimes \mathfrak{o}_n, \mathcal{O})^{\mu_{p^\nu}(\mathfrak{o})}$$

is isomorphic to the \mathfrak{o}_n -module:

$$\{\overline{Q} \in V \mid \overline{Q}(\zeta X) = \overline{Q}(X) \text{ for all } \zeta \in \mu_{p^\nu}(\mathfrak{o}) \text{ and } \overline{Q}(1) = 0\}.$$

Lift \overline{Q} to a Laurent polynomial $Q = \sum_{\mu \in \mathcal{S}} a_\mu X^\mu$ in $\mathfrak{o}[X, X^{-1}]$ where $\mathcal{S} = \{0, \dots, p^\nu - 1\}$.

Then we have:

$$(4) \quad (\zeta^\mu - 1)a_\mu \equiv 0 \pmod{\pi^n} \quad \text{for } \mu \in \mathcal{S} \text{ and } \zeta \in \mu_{p^\nu}(\mathfrak{o})$$

and

$$(5) \quad \sum_{\mu \in \mathcal{S}} a_\mu \equiv 0 \pmod{\pi^n}.$$

For any non-zero μ in \mathcal{S} choose $\zeta \in \mu_{p^\nu}(\mathfrak{o})$ such that $\zeta^\mu \neq 1$. Then $\zeta^\mu - 1$ divides p in \mathfrak{o} and hence (4) implies that $pa_\mu \equiv 0 \pmod{\pi^n}$ for all $\mu \neq 0$. Using (5) it follows that we have $pa_0 \equiv 0 \pmod{\pi^n}$ as well. Hence $pQ \pmod{\pi^n}$ is zero and therefore $p\overline{Q} = 0$ as well. Thus p annihilates the \mathfrak{o}_n -module (3) for all $\nu \geq 1$ and $n \geq 1$. \square

Now assume that $S = R$ and that \mathcal{G}/R is a finite, flat group scheme. Consider the Cartier dual $\mathcal{G}' = \text{spec } \mathcal{A}'$ where $\mathcal{A}' = \text{Hom}_R(\mathcal{A}, R)$. The perfect pairing of finite free \mathfrak{o}_n -modules

$$(6) \quad (\mathcal{A} \otimes \mathfrak{o}_n) \times (\mathcal{A}' \otimes \mathfrak{o}_n) \rightarrow \mathfrak{o}_n$$

induces an isomorphism

$$(7) \quad \text{Ker}((\mathcal{A} \otimes \mathfrak{o}_n)^{\mathcal{G}(\mathfrak{o})} \xrightarrow{\varepsilon} \mathfrak{o}_n) \xrightarrow{\sim} \text{Hom}_{\mathfrak{o}_n}((\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathcal{G}(\mathfrak{o})} / \mathfrak{o}_n, \mathfrak{o}_n).$$

Using (2) it follows that if p^t annihilates $(\mathcal{A}' \otimes \mathfrak{o}_n)_{\mathcal{G}(\mathfrak{o})} / \mathfrak{o}_n$ then p^t annihilates $(\mathcal{A} \otimes \mathfrak{o}_n)^{\mathcal{G}(\mathfrak{o})} / \mathfrak{o}_n$ as well. (The converse is not true in general.)

Hence theorem 1 follows from the next result (applied to the dual p -divisible group).

Theorem 3 *Assume that the p -divisible group G is at most one-dimensional and that the connected-étale exact sequence for G splits over \mathfrak{o} . Then there is an integer $t \geq 1$ such that p^t annihilates the cokernel of the natural map*

$$\mathfrak{o}_n \rightarrow \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})}$$

for all ν and n .

For a finite flat group scheme $\mathcal{G} = \text{spec } \mathcal{A}$ over a ring S , the group

$$\mathcal{G}'(S) = \text{Hom}_{S\text{-alg}}(\text{Hom}_S(\mathcal{A}, S), S) \subset \mathcal{A}$$

consists of the group-like elements in \mathcal{A} i.e. the units a in \mathcal{A} with $\mu(a) = a \otimes a$. In this way $\mathcal{G}'(S)$ becomes a subgroup of the unit group \mathcal{A}^* and hence $\mathcal{G}'(S)$ acts on \mathcal{A} by multiplication. On the other hand $\mathcal{G}'(S)$ acts on \mathcal{G}' by translation, hence on $\mathcal{A}' = \Gamma(\mathcal{G}', \mathcal{O})$ and hence on $\mathcal{A}'' = \mathcal{A}$. Using (1) one checks that the two actions of $\mathcal{G}'(S)$ on \mathcal{A} are the same. This leads to the following description of the cofixed module in theorem 3. Set $A_\nu = \Gamma(G_\nu, \mathcal{O})$ and let J_ν be the ideal in $A_\nu \otimes_R \mathfrak{o}$ generated by the elements $h-1$ with h group-like in this Hopf-algebra over \mathfrak{o} . Thus J_ν is also the \mathfrak{o} -submodule of $A_\nu \otimes_R \mathfrak{o}$ generated by the elements $ha - a$ for $h \in G'_\nu(\mathfrak{o})$ and $a \in A_\nu \otimes_R \mathfrak{o}$. Then we have the formula

$$(8) \quad \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})} = (A_\nu \otimes_R \mathfrak{o}_n) / J_\nu(A_\nu \otimes_R \mathfrak{o}_n) .$$

This implies an isomorphism:

$$(9) \quad \text{Coker}(\mathfrak{o}_n \rightarrow \Gamma(G_\nu \otimes \mathfrak{o}_n, \mathcal{O})_{G'_\nu(\mathfrak{o})}) = \text{Coker}(\mathfrak{o} \rightarrow (A_\nu \otimes_R \mathfrak{o}) / J_\nu) \otimes_{\mathfrak{o}} \mathfrak{o}_n .$$

Hence theorem 3 and therefore also theorem 1 follow from the next claim:

Claim 4 *For a p -divisible group $G = (G_\nu)$ as in theorem 3 there exists an integer $t \geq 1$ such that p^t annihilates the cokernel of the natural map $\mathfrak{o} \rightarrow (A_\nu \otimes_R \mathfrak{o}) / J_\nu$ for all $\nu \geq 1$.*

As a first step in the proof of claim 4 we reduce to the case where G is either étale or connected. For simplicity set $\mathcal{G} = G_\nu \otimes_R \mathfrak{o} = \text{spec } \mathcal{A}$ and define $\mathcal{G}^0, \mathcal{G}^{\text{ét}}, \mathcal{A}^0, \mathcal{A}^{\text{ét}}$ similarly. By assumption we have isomorphisms $\mathcal{G} = \mathcal{G}^0 \times_{\mathfrak{o}} \mathcal{G}^{\text{ét}}$ and $\mathcal{A} = \mathcal{A}^0 \otimes_{\mathfrak{o}} \mathcal{A}^{\text{ét}}$ as group schemes, resp. Hopf-algebras over \mathfrak{o} . There is a compatible splitting of the group-like elements over \mathfrak{o} :

$$\mathcal{G}'(\mathfrak{o}) = \mathcal{G}^{0'}(\mathfrak{o}) \times \mathcal{G}^{\text{ét}'}(\mathfrak{o}) .$$

For elements

$$h^0 \in \mathcal{G}^{0'}(\mathfrak{o}) \subset \mathcal{A}^0 \quad \text{and} \quad h^{\text{ét}} \in \mathcal{G}^{\text{ét}'}(\mathfrak{o}) \subset \mathcal{A}^{\text{ét}}$$

consider the identity:

$$h^0 \otimes h^{\text{ét}} - 1 = h^0 \otimes (h^{\text{ét}} - 1) + (h^0 - 1) \otimes 1 \quad \text{in } \mathcal{A} .$$

It implies that we have

$$J = \mathcal{A}^0 \otimes J^{\text{ét}} + J^0 \otimes \mathcal{A}^{\text{ét}} \quad \text{in } \mathcal{A}$$

where J is the ideal of \mathcal{A} generated by the elements $h - 1$ for $h \in \mathcal{G}'(\mathfrak{o})$ and $J^0, J^{\text{ét}}$ are defined similarly. Hence we have natural surjections

$$\mathcal{A}^0/J^0 \otimes \mathcal{A}^{\text{ét}}/J^{\text{ét}} \rightarrow \mathcal{A}/J$$

and

$$\text{Coker}(\mathfrak{o} \rightarrow \mathcal{A}^0/J^0) \otimes \text{Coker}(\mathfrak{o} \rightarrow \mathcal{A}^{\text{ét}}/J^{\text{ét}}) \rightarrow \text{Coker}(\mathfrak{o} \rightarrow \mathcal{A}/J).$$

Hence it suffices to prove claim 4 in the cases where G is either connected or étale. The étale case is straightforward: We have $G \otimes_R \mathfrak{o} = ((\mathbb{Z}/p^\nu)^h)_{\nu \geq 0}$ where for an abstract group A we denote by \underline{A} the corresponding étale group scheme. Hence $G'_\nu = \mu_{p^\nu}^h$ and $G'_\nu(\mathfrak{o}) = \mu_{p^\nu}(\mathfrak{o})^h$. The inclusion

$$\mu_{p^\nu}(\mathfrak{o})^h = \text{Hom}((\mathbb{Z}/p^\nu)^h, \mathfrak{o}^*) \subset \text{Maps}((\mathbb{Z}/p^\nu)^h, \mathfrak{o}) = A_\nu \otimes \mathfrak{o}$$

identifies $\mu_{p^\nu}(\mathfrak{o})^h$ with the group like elements in $A_\nu \otimes \mathfrak{o}$.

The ideal J_ν of $A_\nu \otimes \mathfrak{o}$ is given by:

$$J_\nu = (\chi_\zeta - 1 \mid \zeta \in \mu_{p^\nu}(\mathfrak{o})^h)$$

where χ_ζ is the character of $(\mathbb{Z}/p^\nu)^h$ defined by the equation

$$\chi_\zeta((a_1, \dots, a_h)) = \zeta_1^{a_1} \cdots \zeta_h^{a_h} \quad \text{where } \zeta = (\zeta_1, \dots, \zeta_h).$$

The functions δ_a for $a \in (\mathbb{Z}/p^\nu)^h$ given by $\delta_a(a) = 1$ and $\delta_a(b) = 0$ if $b \neq a$ generate $A_\nu \otimes \mathfrak{o}$ as an \mathfrak{o} -module. For $a \neq 0$ choose $\zeta \in \mu_{p^\nu}(\mathfrak{o})^h$ with $\zeta^a \neq 1$. Then we have $p = (\zeta^a - 1)\beta$ for some $\beta \in \mathfrak{o}$. Define $f_a \in A_\nu \otimes \mathfrak{o}$ by setting

$$f_a(a) = \beta \quad \text{and} \quad f_a(b) = 0 \text{ for } b \neq a.$$

We then find:

$$f_a(\chi_\zeta - 1) = p\delta_a \quad \text{in } A_\nu \otimes \mathfrak{o}.$$

Hence we have $p\delta_a \in J_\nu$ for all $a \neq 0$ and therefore p annihilates $\text{Coker}(\mathfrak{o} \rightarrow (A_\nu \otimes \mathfrak{o})/J_\nu)$.

The next two sections are devoted to the much more interesting case where G is connected.

3 The connected case I (p -adic Hodge theory)

In this section we reduce the assertion of claim 4 for connected p -divisible groups of arbitrary dimension to an assertion on ideals in $C\langle X_1, \dots, X_d \rangle$. For this reduction we use theorems of Tate in [T].

Thus let $G = (G_\nu)$ be a connected p -divisible group of dimension d over R and set $A = \varprojlim_\nu A_\nu$ where $G_\nu = \text{spec } A_\nu$.

Consider the projective limit $A = \varprojlim A_n$ with the topology inherited from the product topology $\prod A_n$ where the A_n 's are given the π -adic topology. This topology on A is the one defined by the R -submodules $K_n + \pi^k A$ for $n, k \geq 1$ where $K_n = \text{Ker}(A \rightarrow A_n)$. Equivalently it is defined by the spaces $K_n + \pi^n A$ for $n \geq 1$. In [T] section (2.2) it is shown that A is isomorphic to $R[[X_1, \dots, X_d]]$ as a topological R -algebra. If M denotes the maximal ideal of A , then according to [T] Lemma 0 the topology of A coincides with the M -adic topology. Let $A \hat{\otimes}_R \mathfrak{o}$ be the completion of $A \otimes_R \mathfrak{o}$ with respect to the linear topology on $A \otimes_R \mathfrak{o}$ given by the subspaces $M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}$.

Lemma 5 *We have*

$$\varprojlim (A_n \otimes_R \mathfrak{o}) = A \hat{\otimes}_R \mathfrak{o} = \mathfrak{o}[[X_1, \dots, X_d]]$$

as topological rings.

Proof Consider the isomorphisms

$$\begin{aligned} \varprojlim_n (A_n \otimes_R \mathfrak{o}) &= \varprojlim_n (A_n \otimes_R (\varprojlim_k \mathfrak{o} / \pi^k \mathfrak{o})) \\ &\stackrel{(1)}{=} \varprojlim_n \varprojlim_k (A_n \otimes_R \mathfrak{o} / \pi^k \mathfrak{o}) \\ &= \varprojlim_n \varprojlim_k (A \otimes_R \mathfrak{o}) / ((K_n + \pi^k A) \otimes_R \mathfrak{o} + A \otimes_R \pi^k \mathfrak{o}) \\ &\stackrel{(2)}{=} \varprojlim_n (A \otimes_R \mathfrak{o}) / ((K_n + \pi^n A) \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}) \\ &\stackrel{(3)}{=} \varprojlim_n (A \otimes_R \mathfrak{o}) / (M^n \otimes_R \mathfrak{o} + A \otimes_R \pi^n \mathfrak{o}) \\ &= A \hat{\otimes}_R \mathfrak{o} \\ &\stackrel{(4)}{=} \mathfrak{o}[[X_1, \dots, X_d]]. \end{aligned}$$

Here (1) holds because \varprojlim commutes with finite direct sums, (2) is true by cofinality, (3) holds because the topology on A can also be described as the M -adic topology. Finally (4) follows from the definition of $A \hat{\otimes}_R \mathfrak{o}$ and the fact that $A = R[[X_1, \dots, X_d]]$. \square

The \mathfrak{o} -algebra $A \hat{\otimes}_R \mathfrak{o} = \varprojlim_{\nu} (A_{\nu} \otimes_R \mathfrak{o})$ contains the ideal $\tilde{J} = \varprojlim_{\nu} J_{\nu}$.

Claim 6 *We have*

$$A \hat{\otimes}_R \mathfrak{o} / (\mathfrak{o} + \tilde{J}) = \varprojlim_{\nu} (A_{\nu} \otimes_R \mathfrak{o} / (\mathfrak{o} + J_{\nu})) .$$

Proof The inclusion $G_{\nu} \subset G_{\nu+1}$ corresponds to a surjection of Hopf-algebras $A_{\nu+1} \rightarrow A_{\nu}$. Hence $A_{\nu+1} \otimes_R \mathfrak{o} \rightarrow A_{\nu} \otimes_R \mathfrak{o}$ is surjective as well and group-like elements are mapped to group-like elements. The map on group-like elements is surjective because it corresponds to the surjective map $G'_{\nu+1}(\mathfrak{o}) \rightarrow G'_{\nu}(\mathfrak{o})$. Note here that $G'_{\mu}(\mathfrak{o}) = G'_{\mu}(C)$ for all μ . It follows that the map $J_{\nu+1} \rightarrow J_{\nu}$ is surjective as well. In the exact sequence of projective systems

$$0 \rightarrow (\mathfrak{o} + J_{\nu}) \rightarrow (A_{\nu} \otimes_R \mathfrak{o}) \rightarrow (A_{\nu} \otimes_R \mathfrak{o} / (\mathfrak{o} + J_{\nu})) \rightarrow 0$$

the system $(\mathfrak{o} + J_{\nu})$ is therefore Mittag-Leffler. Hence the sequence of projective limits is exact and the claim follows because the sum $\mathfrak{o} + J_{\nu}$ is direct: Group-like elements of $A_{\nu} \otimes_R \mathfrak{o}$ are mapped to 1 by the counit ε_{ν} . Therefore we have

$$(10) \quad J_{\nu} \subset I_{\nu} := \text{Ker}(\varepsilon_{\nu} : A_{\nu} \otimes_R \mathfrak{o} \rightarrow \mathfrak{o}) .$$

The sum $\mathfrak{o} + I_{\nu}$ being direct we are done. \square

Because of claim 6 and the surjectivity of the maps $A_{\nu+1} \otimes_R \mathfrak{o} \rightarrow A_{\nu} \otimes_R \mathfrak{o}$, claim 4 for connected groups is equivalent to the next assertion:

Claim 7 *Let G be a connected p -divisible group with $\dim G \leq 1$. Then there is some $t \geq 1$ such that p^t annihilates*

$$\text{Coker}(\mathfrak{o} \rightarrow A \hat{\otimes}_R \mathfrak{o} / \tilde{J}) .$$

For connected G of arbitrary dimension consider the Tate module of G'

$$TG' = \varprojlim_{\nu} G'_{\nu}(C) = \varprojlim_{\nu} G'_{\nu}(\mathfrak{o}) \subset \varprojlim_{\nu} (A_{\nu} \otimes_R \mathfrak{o}) = A \hat{\otimes}_R \mathfrak{o} .$$

Let J be the ideal of $A \hat{\otimes}_R \mathfrak{o}$ generated by the elements $h - 1$ for $h \in TG'$. The image of J under the reduction map $A \hat{\otimes}_R \mathfrak{o} \rightarrow A_{\nu} \otimes_R \mathfrak{o}$ lies in J_{ν} . It follows that $J \subset \tilde{J}$. With I_{ν} as in (10) we set $I = \varprojlim_{\nu} I_{\nu}$, an ideal in $A \hat{\otimes}_R \mathfrak{o}$. We have $J \subset \tilde{J} \subset I$ because of (10). Since $A \hat{\otimes}_R \mathfrak{o} = \mathfrak{o} \oplus I$, we get a surjection

$$(11) \quad I/J \rightarrow \text{Coker}(\mathfrak{o} \rightarrow A \hat{\otimes}_R \mathfrak{o} / \tilde{J}) .$$

Thus claim 7 will be proved if we can show that $p^t I \subset J$ at least for $\dim G = 1$. The construction in [T] section (2.2) shows that under the isomorphism of \mathfrak{o} -algebras

$$A \hat{\otimes}_R \mathfrak{o} = \mathfrak{o}[[X_1, \dots, X_d]] \quad \text{we have } I = (X_1, \dots, X_d) .$$

We will view the elements of $A \hat{\otimes}_R \mathfrak{o}$ and in particular those of J as analytic functions on the open d -dimensional polydisc

$$\Delta^d = \{x \in C^d \mid |x_i| < 1 \text{ for all } i\} .$$

Because of the inclusion $J \subset I$ all functions in J vanish at $0 \in \Delta^d$. There are no other common zeroes:

Proposition 8 (Tate) *The zero set of J in Δ^d consists only of the origin $o \in \Delta^d$.*

Proof The \mathfrak{o} -valued points of the p -divisible group G ,

$$G(\mathfrak{o}) = \varprojlim_i \varinjlim_{\nu} G_{\nu}(\mathfrak{o}/\pi^i \mathfrak{o})$$

can be identified with continuous \mathfrak{o} -algebra homomorphisms

$$G(\mathfrak{o}) = \text{Hom}_{\text{cont,alg}}(A, \mathfrak{o}) = \text{Hom}_{\text{cont,alg}}(A \hat{\otimes}_R \mathfrak{o}, \mathfrak{o}) .$$

Moreover we have a homeomorphism

$$\Delta^d \xrightarrow{\sim} G(\mathfrak{o}) \quad \text{via} \quad x \mapsto (f \mapsto f(x)) .$$

Here $f \in A \hat{\otimes}_R \mathfrak{o}$ is viewed as a formal power series over \mathfrak{o} . The group structure on $G(\mathfrak{o})$ induces a Lie group structure on Δ^d with $0 \in \Delta^d$ corresponding to $1 \in G(\mathfrak{o})$. Let U be the group of 1-units in \mathfrak{o} . Proposition 11 of [T] asserts that the homomorphism of Lie groups

$$(12) \quad \alpha : \Delta^d = G(\mathfrak{o}) \rightarrow \text{Hom}_{\text{cont}}(TG', U), \quad x \mapsto (h \mapsto h(x))$$

is *injective*. Note here that $TG' \subset A \hat{\otimes}_R \mathfrak{o}$. Let $x \in \Delta^d$ be a point in the zero set of J . Then we have $(h-1)(x) = 0$ i.e. $h(x) = 1$ for all $h \in TG'$. Hence x maps to $1 \in \text{Hom}_{\text{cont}}(TG', U)$. Since α is injective, it follows that we have $x = 0$. \square

If a Hilbert Nullstellensatz were true in $C\langle X_1, \dots, X_d \rangle$ we could conclude that we had $\sqrt{J \otimes C} = I \otimes C$ and with further arguments from [T] we would get $p^t I \subset J$. However the Nullstellensatz does not hold in the ring $C\langle X_1, \dots, X_d \rangle$.

In the next section we will provide a replacement which is proved for $d = 1$ and conjectured for $d \geq 2$. In order to apply it to the ideal $J \otimes C$ in $C\langle X_1, \dots, X_d \rangle$ we need to know the following assertion which is stronger than proposition 8. For $x \in C^m$ set $\|x\| = \max_i |x_i|$.

Proposition 9 *Let h_1, \dots, h_r be a \mathbb{Z}_p -basis of $TG' \subset \mathfrak{o}[[X_1, \dots, X_d]]$ and set $H(x) = (h_1(x), \dots, h_r(x))$ and $\mathbf{1} = (1, \dots, 1)$. Then there is a constant $\delta > 0$ such that we have:*

$$\|H(x) - \mathbf{1}\| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.$$

Proof The \mathbb{Z}_p -rank r of TG' is the height of G' and hence we have $r \geq d = \dim G$. Consider the following diagram (*) on p. 177 of [T]:

$$\begin{array}{ccccccc} 1 & \longrightarrow & G(\mathfrak{o})_{\text{tors}} & \longrightarrow & G(\mathfrak{o}) & \xrightarrow{L} & t_G(C) \longrightarrow 0 \\ & & \alpha_0 \downarrow \wr & & \alpha \downarrow & & d\alpha \downarrow \\ 1 & \longrightarrow & \text{Hom}(TG', U_{\text{tors}}) & \longrightarrow & \text{Hom}(TG', U) & \xrightarrow{\log_*} & \text{Hom}(TG', C) \longrightarrow 0. \end{array}$$

Here the Hom-groups refer to continuous homomorphisms and the map α was defined in equation (12) above. The map L is the logarithm map to the tangent space $t_G(C)$ of G and \log_* is induced by $\log : U \rightarrow C$. According to [T] proposition 11 the maps α and $d\alpha$ are injective and α_0 is bijective. It will suffice to prove the following two statements:

I) For any $\varepsilon > 0$ there is a constant $\delta(\varepsilon) > 0$ such that

$$\|H(x) - \mathbf{1}\| \geq \delta(\varepsilon) \text{ for all } x \in \Delta^d \text{ with } \|x\| \geq \varepsilon .$$

II) There are $\varepsilon > 0$ and $a > 0$ such that

$$\|H(x) - \mathbf{1}\| \geq a\|x\| \text{ for all } x \in \Delta^d \text{ with } \|x\| \leq \varepsilon .$$

Identifying $G(\mathfrak{o})$ with Δ^d where we write the induced group structure on Δ^d as \oplus , and identifying TG' with \mathbb{Z}_p^r via the choice of the basis h_1, \dots, h_r , the above diagram becomes the following one where $A = dH$ and H_0 is the restriction of H to $(\Delta^d)_{\text{tors}}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\Delta^d)_{\text{tors}} & \longrightarrow & \Delta^d & \xrightarrow{L} & C^d \longrightarrow 0 \\ & & \downarrow H_0 & & \downarrow H & & \downarrow A=dH \\ 1 & \longrightarrow & U_{\text{tors}}^r & \longrightarrow & U^r & \xrightarrow{\log} & C^r \longrightarrow 0. \end{array}$$

Assume that assertion I is wrong for some $\varepsilon > 0$. Then there is a sequence $x^{(i)}$ of points in Δ^d with $\|x^{(i)}\| \geq \varepsilon$ such that $H(x^{(i)}) \rightarrow \mathbf{1}$ for $i \rightarrow \infty$. It follows that $A(L(x^{(i)})) = \log H(x^{(i)}) \rightarrow 0$ for $i \rightarrow \infty$. Since A is an injective linear map between finite dimensional C -vector spaces, there exists a constant $a > 0$ such that we have

$$(13) \quad \|A(v)\| \geq a\|v\| \quad \text{for all } v \in C^d .$$

Hence we see that $L(x^{(i)}) \rightarrow 0$ for $i \rightarrow \infty$. Since L is a local homeomorphism, there exists a sequence $y^{(i)} \rightarrow 0$ in Δ^d with $L(x^{(i)}) = L(y^{(i)})$ for all i . The sequence $z^{(i)} = x^{(i)} \ominus y^{(i)}$ in Δ^d satisfies $L(z^{(i)}) = 0$ and hence lies in $(\Delta^d)_{\text{tors}}$. We have $H_0(z^{(i)}) = H(x^{(i)})H(y^{(i)})^{-1}$. Moreover $H(x^{(i)}) \rightarrow \mathbf{1}$ by assumption and $H(y^{(i)}) \rightarrow \mathbf{1}$ since $y^{(i)} \rightarrow 0$. Hence $H_0(z^{(i)}) \rightarrow \mathbf{1}$ and therefore $H_0(z^{(i)}) = \mathbf{1}$ for all $i \gg 0$ since the subspace topology on $U_{\text{tors}} \subset U$ is the discrete topology. The map H_0 being bijective we find that $z^{(i)} = 0$ for $i \gg 0$ and therefore $x^{(i)} = y^{(i)}$ for $i \gg 0$. This implies that $x^{(i)} \rightarrow 0$ for $i \rightarrow \infty$ contradicting the assumption $\|x^{(i)}\| \geq \varepsilon$ for all i . Hence assertion I) is proved.

We now turn to assertion II). Set $X = (X_1, \dots, X_d)$. Then we have

$$H(X) = \mathbf{1} + AX + (\deg \geq 2) .$$

Componentwise this gives for $1 \leq j \leq r$

$$h_j(x) - 1 = \sum_{i=1}^d a_{ij}x_i + (\deg \geq 2)_j .$$

Let a be the constant from equation (13) and choose $\varepsilon > 0$, such that for $\|x\| \leq \varepsilon$ we have

$$\|(\deg \geq 2)_j\| \leq \frac{a}{2}\|x\| \quad \text{for } 1 \leq j \leq r .$$

For any x with $\|x\| < \varepsilon$, according to (13) there is an index j with

$$\left| \sum_{i=1}^d a_{ij}x_i \right| \geq a\|x\| .$$

This implies that we have

$$|h_j(x) - 1| = \left| \sum_{i=1}^d a_{ij}x_i + (\deg \geq 2)_j \right| = \left| \sum_{i=1}^d a_{ij}x_i \right| \geq a\|x\|$$

and hence

$$\|H(x) - \mathbf{1}\| \geq a\|x\| .$$

□

4 The connected case II (the p -adic Corona problem)

As remarked in the previous section we need a version of the Hilbert Nullstellensatz in $C\langle X_1, \dots, X_d \rangle$ for the case where the zero set is $\{0\} \subset \Delta^d$. The only result for $C\langle X_1, \dots, X_d \rangle$ in the spirit of the Nullstellensatz that I am aware of concerns an empty zero set:

P -adic Corona theorem 10 (van der Put, Bartenwerfer) *For f_1, \dots, f_n in $C\langle X_1, \dots, X_d \rangle$ the following conditions are equivalent:*

- 1) The functions f_1, \dots, f_n generate the C -algebra $C\langle X_1, \dots, X_d \rangle$.
- 2) There is a constant $\delta > 0$ such that

$$\max_{1 \leq j \leq n} |f_j(x)| \geq \delta \quad \text{for all } x \in \Delta^d.$$

It is clear that the first condition implies the second. The non-trivial implication was proved by van der Put for $d = 1$ in [P] and by Bartenwerfer in general, c.f. [B]. Both authors give a more precise statement of the theorem where the norms of possible functions g_j with $\sum_j f_j g_j = 1$ are estimated.

Consider the following conjecture which deals with the case where the zero set may contain $\{0\}$.

Conjecture 11 *For g_1, \dots, g_n in $C\langle X_1, \dots, X_d \rangle$ the following conditions are equivalent:*

- 1) $(g_1, \dots, g_n) \supset (X_1, \dots, X_d)$.
- 2) There is a constant $\delta > 0$ such that

$$(14) \quad \max_{1 \leq j \leq n} |g_j(x)| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.$$

As above, immediate estimates show that the first condition implies the second. Note also that if some g_j does not vanish at $x = 0$ we have

$$\max_{1 \leq j \leq n} |g_j(x)| \geq \delta' > 0 \text{ in a neighborhood of } x = 0.$$

Together with (14) this implies that

$$\max_{1 \leq j \leq n} |g_j(x)| \geq \delta'' > 0 \quad \text{for all } x \in \Delta^d.$$

The p -adic Corona theorem then gives $(g_1, \dots, g_n) = C\langle X_1, \dots, X_d \rangle$. Thus condition 1 follows in this case.

Proposition 12 *The preceding conjecture is true for $d = 1$.*

Proof As explained above, we may assume that all functions g_1, \dots, g_n vanish at $x = 0$. Then $f_j(X) = X^{-1}g_j(X)$ is in $C\langle X \rangle$ for every $1 \leq j \leq n$ and estimate (14) implies the estimate

$$\max_{1 \leq j \leq n} |f_j(x)| \geq \delta \quad \text{for all } x \in \Delta^1.$$

The p -adic Corona theorem for $d = 1$ now shows that

$$(f_1, \dots, f_n) = (1) \quad \text{and hence} \quad (g_1, \dots, g_n) = (X).$$

□

Let us now return to p -divisible groups and recall the surjection (11):

$$(15) \quad I/J \twoheadrightarrow \text{Coker}(\mathfrak{o} \rightarrow A \hat{\otimes}_R \mathfrak{o} / \tilde{J}).$$

Here $I = (X_1, \dots, X_d)$ in $\mathfrak{o}[[X_1, \dots, X_d]]$ and J is the ideal generated by the elements $h - 1$ for $h \in TG'$. Let $J_0 \subset J$ be the ideal generated by the elements $h_1 - 1, \dots, h_r - 1$ where h_1, \dots, h_r form a \mathbb{Z}_p -basis of TG' . In proposition 9 we have seen that for some $\delta > 0$ we have

$$\max_{1 \leq j \leq r} |h_j(x) - 1| \geq \delta \|x\| \quad \text{for all } x \in \Delta^d.$$

Conjecture 11 (which is true for $d = 1$) would therefore imply

$$(h_1 - 1, \dots, h_r - 1) = (X_1, \dots, X_d) \quad \text{in } C\langle X_1, \dots, X_d \rangle.$$

Thus we would find some $t \geq 1$, such that we have

$$p^t X_i \in J_0 \subset \mathfrak{o}[[X_1, \dots, X_d]] \quad \text{for all } 1 \leq i \leq r$$

and hence also $p^t I \subset J_0 \subset J$. Using the surjection (15) this would prove claim 7 and hence theorem 3 without restriction on $\dim G$. Also theorem 1 would follow without restriction on $\dim G'$. As it is we have to assume $\dim G \leq 1$ resp. $\dim G' \leq 1$ in these assertions.

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